The flow equation approach to the pairing instability problem

T. Domański, A. Donabidowicz
Institute of Physics, M. Curie Skłodowska University, 20-031 Lublin, Poland

(Dated: February 6, 2008)

By means of the continuous unitary transformation similar to a general scheme of the Renormalization Group (RG) procedure we study the issue of symmetry breaking and pairing instability in the system of interacting fermions. Constructing a generalized version of the Bogoliubov transformation we show that formation of the fermion pairs and their superconductivity/superfluidity can appear at different temperatures. It is shown that strong quantum fluctuations can destroy the long-range order without breaking the fermion pairs which may still exist as incoherent and/or damped entities. Such unusual phase is characterized by a partial suppression of the density of states near the Fermi energy and by residual collective features like the sound-wave mode in the fermion pair spectrum.

Formation of the fermion pairs is a common phenomenon for various physical systems of the interacting particles such as: electrons, nucleons, atoms and quarks. Binding energy and the spatial extent of fermion pairs may vary from case to case depending on particular species and on specific interaction mechanism. To give some examples let us mention that pairing can be driven by:

(i) exchange of phonons (in classical superconductors, MgB_2, etc),

(ii) exchange of magnons (superconductivity of the heavy fermion compounds),

(iii) strong correlations (the high T_c superconductors),

(iv) Feshbach resonance (superfluidity of the ultracold fermion atoms),

(v) or by other effects (nucleon pairing in nuclei, superfuidity of the neutron stars).

Usually formation of the fermion pairs goes hand in hand with appearance of the order parameter which consequently leads either to superconductivity (for charged particles such the conduction band electrons or holes) or to superfluidity (for electrically neutral objects like ^{3}He or the ultracold fermion atoms in magnetooptical traps). However, a simultaneous formation of pairs and emergence of the symmetry broken phases needs not be a rule. We will show here example that both these phenomena are distinct and happen to coincide at the same critical temperature T_c only when the quantum fluctuations are weak.

I. HAMILTONIAN OF THE INTERACTING FERMIONS

System of the interacting fermions can be described by the following Hamiltonian

$$\hat{H} = \sum_{\mathbf{k},\sigma} (\epsilon_{\mathbf{k}} - \mu) \hat{c}^\dagger_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} + \frac{1}{2} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \sum_{\sigma,\sigma'} U_{\mathbf{k},\mathbf{k}'}(\mathbf{q}) \hat{c}^\dagger_{\mathbf{k}+\frac{\mathbf{q}}{2},\sigma} \hat{c}^\dagger_{\mathbf{k}'-\frac{\mathbf{q}}{2},\sigma'} \hat{c}_{\mathbf{k}'+\frac{\mathbf{q}}{2},\sigma'} \hat{c}_{\mathbf{k}-\frac{\mathbf{q}}{2},\sigma} \tag{1}$$

where $\varepsilon_{\mathbf{k}}$ is a single particle energy for a given momentum $\mathbf{k}$ and σ corresponds to additional quantum numbers like for instance spin $\uparrow, \downarrow$ for electrons, the angular momentum for atoms or the isospin for nucleons. The two-body interactions are described by the second term with the potential $U_{\mathbf{k},\mathbf{k}'}(\mathbf{q})$. We use in (1) the standard notation for the creation (annihilation) operators $\hat{c}^\dagger_{\mathbf{k}\sigma}$ ($\hat{c}_{\mathbf{k}\sigma}$).

In general there can arise various kinds of ordering, for instance: ferromagnetism, antiferromagnetism, charge ordering, superconducting BCS state, etc. We will focus here on the pairing instabilities. For this purpose we further consider the Hamiltonian reduced only to $\mathbf{q}=\mathbf{0}$ channel

$$\hat{H} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} \hat{c}^\dagger_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \; \hat{c}^\dagger_{\mathbf{k}\uparrow} \hat{c}^\dagger_{-\mathbf{k}\downarrow} \; \hat{c}_{-\mathbf{k}'\downarrow} \hat{c}_{\mathbf{k}'\uparrow} \tag{2}$$

with $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$ and we assume the two-body potential to be attractive $V_{\mathbf{k},\mathbf{k}'} < 0$. We will investigate this reduced BCS Hamiltonian using the nonperturbative method (described in section III) which belong to a family of the Renormalization Group techniques [1].

II. RENORMALIZATION GROUP APPROACH

Thermodynamics of the system (total energy, specific heat, pressure, etc) can be computed from the partition function $\mathcal{Z} = \mathrm{Tr} e^{-\hat{H}/k_B T}$. It is convenient to express $\mathcal{Z}$ using the path integrals over the Grassmann variables $\psi_{\mathbf{k},\sigma}$, $\psi^*_{\mathbf{k},\sigma}$ (which formally represent the eigenvalues of the annihilation $\hat{c}_{\mathbf{k},\sigma}$ and creation $\hat{c}^\dagger_{\mathbf{k},\sigma}$ operators)

$$\mathcal{Z} = \int D[\psi, \psi^*] \; e^{-S}. \tag{3}$$

The action consists of two contributions $S = S_0 + S_I$ where the quadratic term

$$S_0 = \sum_{\sigma} \int_k \psi^*_{\mathbf{k},\sigma} \; (i\omega_n - \xi_{\mathbf{k}}) \; \psi_{\mathbf{k},\sigma} \tag{4}$$

corresponds to a free part and integration in the equation (4) runs over the four-vector $k \equiv (i\omega_n, \mathbf{k})$ with the

fermion Matsubara frequencies $\omega_n = (2n+1)\pi k_B T$. The second quartic term refers to the two-body interactions

$$S_I = -\int_{k,k'} V_{\mathbf{k},\mathbf{k}'} \psi^*_{\mathbf{k},\uparrow} \psi^*_{-\mathbf{k},\downarrow} \psi_{-\mathbf{k}',\downarrow} \psi_{\mathbf{k}'\uparrow} \tag{5}$$

As far as the dynamic quantities are concerned (various correlation functions) they are derivable directly from the generating functional

$$\mathcal{G}[\chi,\chi^*] = \log\left[\mathcal{Z}^{-1}\sum_\sigma \int D[\psi,\psi^*] e^{-(S+\int_k \psi^*_{\mathbf{k},\sigma}\chi_{\mathbf{k},\sigma}+\psi_{\mathbf{k},\sigma}\chi^*_{\mathbf{k},\sigma})}\right] \tag{6}$$

where $\chi_{\mathbf{k},\sigma}$ and $\chi^*_{\mathbf{k},\sigma}$ are the Grassman source fields. For instance, the single particle excitations can be determined via the two-point Green's function $\frac{\delta}{\delta\chi^*_{\mathbf{k},\sigma}} \frac{\delta}{\delta\chi_{\mathbf{k},\sigma}} \mathcal{G}[\chi,\chi^*]_{\chi=0,\chi^*=0}$.

Physical properties of the system under consideration depend predominantly on such fermion states which are located near the Fermi surface. Other states distant from the Fermi energy are less relevant therefore it is useful to make a distinction between their contributions to the partition function

$$\mathcal{Z} = \int D^{<\Lambda}[\psi,\psi^*] \int D^{>\Lambda}[\psi,\psi^*]\ e^{S[\psi,\psi^*]} \tag{7}$$

where symbol $D^{<\Lambda}[\psi,\psi^*]$ corresponds to the fermion states whose distance from the Fermi energy is smaller than a given cut-off $|\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}_F}| < \Lambda$. In the numerical RG method [1] one first integrates out the high energy fermion states. After completing such integration one is left with only the low energy states. Partition function is then $\mathcal{Z} = \int D^{<\Lambda}[\psi,\psi^*]\ e^{-S^\Lambda[\psi,\psi^*]}$ where the renormalized action is defined by

$$e^{-S^\Lambda[\psi,\psi^*]} = \int D^{>\Lambda}[\psi,\psi^*]\ e^{-S[\psi,\psi^*]}. \tag{8}$$

This action (8) can be then cast into (6) simplifying the calculation of the generating functional

$$\mathcal{G}[\chi,\chi^*] = \log\left[\mathcal{Z}^{-1}\int D^{<\Lambda}[\psi,\psi^*] e^{-(S^\Lambda+\int_k^{<\Lambda} \psi^*_{\mathbf{k}}\chi_{\mathbf{k}}+\psi_{\mathbf{k}}\chi^*_{\mathbf{k}})}.\right] \tag{9}$$

Such idea of *mode elimination* has been introduced in the theoretical physics a long time ago [2]. Wilson adopted it to the solid state physics by proposing a sequential integration of the fermion fields down to some small cut-off Λ. Reducing bit by bit the cut-off Λ to infinitesimally small values he was able to study emergence of the critical phenomena [1]. For a more specific discussion of various RG formulations we recommend the the review papers [3].

However, in the case of symmetry broken phases (such as the superconducting state) the simple scaling procedure usually fails due to a natural lower boundary cut-off (the energy gap Δ of the single particle excitations). It has been even claimed that conventional RG techniques are blind with respect to the symmetry-broken states [4]. The situation is not that much hopeless, there are possible routes to circumvent this problem. Let us mention two of them

(i) one can impose by hand an infinitesimal symmetry-breaking parameter $\Delta(\Lambda)$ at a certain initial condition $\Lambda = \Lambda_0$ and then its physical meaning would eventually establish upon the flow $\Lambda \to 0$ [5],

(ii) in order to eliminate the interacting part S_I one can enlarge the Hilbert space by introducing the linear coupling to the boson fields Φ, Φ^* via the Hubbard - Stratonovich transformation [6].

Using the first procedure one must be cautious how to constrain the small symmetry breaking parameter while in the latter method there rise additional complications dealing with the new boson fields. Perhaps the second option is more natural because after all the interactions are always mediated by certain boson fields (phonons, photons, gravitons, etc). Unfortunately in practice it is hard to go beyond a simple saddle point approximation except than by taking into account the small Gaussian corrections around it. In the next section we shall briefly describe a more convenient procedure which has been proposed in 1994.

III. CONTINUOUS CANONICAL TRANSFORMATION

An alternative approach to deal with the many-body effects (which in particular is suitable for studying the symmetry broken phases) has been invented by Wegner [7] and independently by Wilson and Głazek [8]. Instead of integrating out the high energy states (fast modes) one constructs a continuous unitary transformation $\hat{H}(l) = \hat{U}(l)\hat{H}\hat{U}^\dagger(l)$ with a purpose to simplify the Hamiltonian either to diagonal or at least to a block-diagonal form. This is achieved through a series of infinitesimal steps $\hat{H} \longrightarrow \hat{H}(l) \longrightarrow \hat{H}(\infty)$, where l stands for some formal continuous parameter. In a course of transformation the Hamiltonian evolves according to the differential *flow equation*

$$\partial_l \hat{H}(l) = \left[\hat{\eta}(l), \hat{H}(l)\right] \tag{10}$$

where the generating operator is defined as $\hat{\eta}(l) = -\hat{U}(l)\ \partial_l \hat{U}^\dagger(l)$.

This method has a similarity to the traditional RG scaling procedure because

(i) diagonalization of the high energy sector occurs mainly during initial part

(ii) while the low energy states are worked out at the very end of transformation.

Roughly speaking, a role of the Wilson's cut-off energy Λ is played here by $\Lambda^l = \frac{1}{\sqrt{l}}$.

Advantage of such new procedure becomes particularly clear when investigating the mutual relations between the high and small energy states (i.e. between the *fast* and *small* modes). In the continuous canonical transformation one treats both energy sectors on equal footing throughout the whole transformation process. Thereof their feedback effects are feasible to study.

For the Hamiltonians $\hat{H} = \hat{H}_0 + \hat{H}_1$ (where H_0 is a diagonal part in a given representation) it has been proposed [7] to choose

$$\hat{\eta}(l) = \left[\hat{H}_0(l), \hat{H}_1(l)\right] \tag{11}$$

so that the off-diagonal term is eliminated at the asymptotic point

$$\lim_{l\to\infty} \hat{H}_1(l) = 0. \tag{12}$$

There are also other possible ways for constructing the generating operator $\hat{\eta}$. Some a survey we recommend the recent review papers [9, 10].

As far as the diagonalization is concerned the continuous canonical transformation turns out to be a rather convenient tool. However, if one needs the correlation functions then the situation becomes more cumbersome. For example, to determine the correlation functions $\langle \hat{A}(t)\hat{B}(t')\rangle$ one needs the statistical average

$$\begin{aligned}\mathrm{Tr}\left\{e^{-\beta\hat{H}}\hat{O}\right\} &= \mathrm{Tr}\left\{\hat{U}(l)e^{-\beta\hat{H}}\hat{O}\hat{U}^\dagger(l)\right\} \\ &= \mathrm{Tr}\left\{e^{-\beta\hat{H}(l)}\hat{O}(l)\right\}\end{aligned} \tag{13}$$

where $\hat{O}(l) = \hat{U}(l)\hat{O}\hat{U}^\dagger(l)$. The easiest way to carry out the statistical averaging (13) is in the limit $l \longrightarrow \infty$ when $\hat{H}(\infty)$ becomes (block-)diagonal. However, this requires a simultaneous transformation of the observables $\hat{O} \longrightarrow \hat{O}(l) \longrightarrow \hat{O}(\infty)$. The corresponding flow equation is given in a familiar form

$$\partial_l \hat{O}(l) = \left[\hat{\eta}(l), \hat{O}(l)\right]. \tag{14}$$

Thus, calculation of the correlation functions is rather more relative to the projection techniques.

IV. THE BILINEAR HAMILTONIAN

To illustrate how the flow equation method actually works we first consider the bilinear Hamiltonian

$$\hat{H} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} \hat{c}^\dagger_{\mathbf{k}\sigma}\hat{c}_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left(\Delta_{\mathbf{k}}\, \hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow} + \Delta^*_{\mathbf{k}}\, \hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\uparrow}\right) \tag{15}$$

which can be solved exactly, in particular by a single step Bogoliubov transformation [11]. The off-diagonal terms in the Hamiltonian (15) can be thought as resulting from the mean field approximation for the weak pairing potential $V_{\mathbf{k},\mathbf{k}'} < 0$ with a usual definition of the order parameter $\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'}\langle \hat{c}_{-\mathbf{k}'\downarrow}\hat{c}_{\mathbf{k}'\uparrow}\rangle$.

To get the rigorous solution we now construct a continuous transformation such that the convoluted states $|\mathbf{k},\uparrow>$ and $|-\mathbf{k},\downarrow>$ will be decoupled. This continuous process depends on a distance from the Fermi surface $|\xi_{\mathbf{k}}|$ (and the same holds for computation of the coherence factors $u_{\mathbf{k}}$, $v_{\mathbf{k}}$).

Using the Wegner's proposal (11) we obtain the two coupled flow equations

$$\partial_l\, \xi_{\mathbf{k}}(l) = 4\xi_{\mathbf{k}}(l)|\Delta_{\mathbf{k}}(l)|^2 \tag{16}$$

$$\partial_l\, \Delta_{\mathbf{k}}(l) = -4(\xi_{\mathbf{k}}(l))^2\Delta^*_{\mathbf{k}}(l) \tag{17}$$

for l-dependent quantities $\xi_{\mathbf{k}}(l)$ and $\Delta_{\mathbf{k}}(l)$. The second equation (17) yields

$$|\Delta_{\mathbf{k}}(l)| = |\Delta_{\mathbf{k}}|\exp\left\{-4\int_0^l dl'[\xi_{\mathbf{k}}(l')]^2\right\} \tag{18}$$

which proves that indeed the off-diagonal terms gradually diminish under the flow $l\to\infty$. There is a singular point $\mathbf{k}=\mathbf{k}_F$ which is unaffected by the transformation but in the thermodynamic limit (i.e. for a macroscopic number of particles N) its role becomes marginal.

By combining the equations (16,17) one gets the following invariance

$$\partial_l\left\{(\xi_{\mathbf{k}}(l))^2 + |\Delta_{\mathbf{k}}(l)|^2\right\} = 0. \tag{19}$$

which implies that in the limit $l\to\infty$ the eigenvalues have a character of the Bogoliubov spectrum [12, 13]

$$\xi_{\mathbf{k}}(\infty) = \mathrm{sgn}(\xi_{\mathbf{k}})\sqrt{(\xi_{\mathbf{k}})^2 + |\Delta_{\mathbf{k}}|^2}. \tag{20}$$

In figure 1 we illustrate evolution of the parameter $\Delta_{\mathbf{k}}(l)$ which initially was assumed to be constant $\Delta_{\mathbf{k}} = \Delta$. In the first turn disappearance of $\Delta_{\mathbf{k}}(l)$ occurs for states distant from the Fermi energy and then in the second turn to states located nearby to μ. This evolution is accompanied by the renormalization of fermion energies $\xi_{\mathbf{k}}(l)$. Finally for $l\to\infty$ the quasiparticle energies evolve to the gaped Bogoliubov dispersion (20).

In order to specify the complete single particle spectrum we now derive the correlation function $\langle \hat{c}_{\mathbf{k}\sigma}(t)\hat{c}^\dagger_{\mathbf{k}\sigma}(t')\rangle$ where time evolution is given by the standard relation $\hat{O}(t) = e^{it\hat{H}}\hat{O}e^{-it\hat{H}}$. As emphasized in the previous section this requires determination of the l-dependent single particle operators. From a detailed analysis [12] we obtain the Bogoliubov-type parameterizations

$$\hat{c}_{\mathbf{k}\uparrow}(l) = u_{\mathbf{k}}(l)\hat{c}_{\mathbf{k}\uparrow} + v_{\mathbf{k}}(l)\hat{c}^\dagger_{-\mathbf{k}\downarrow} \tag{21}$$

$$\hat{c}_{-\mathbf{k}\downarrow}(l)^\dagger = -v_{\mathbf{k}}(l)\hat{c}_{\mathbf{k}\uparrow} + u_{\mathbf{k}}(l)\hat{c}^\dagger_{-\mathbf{k}\downarrow} \tag{22}$$

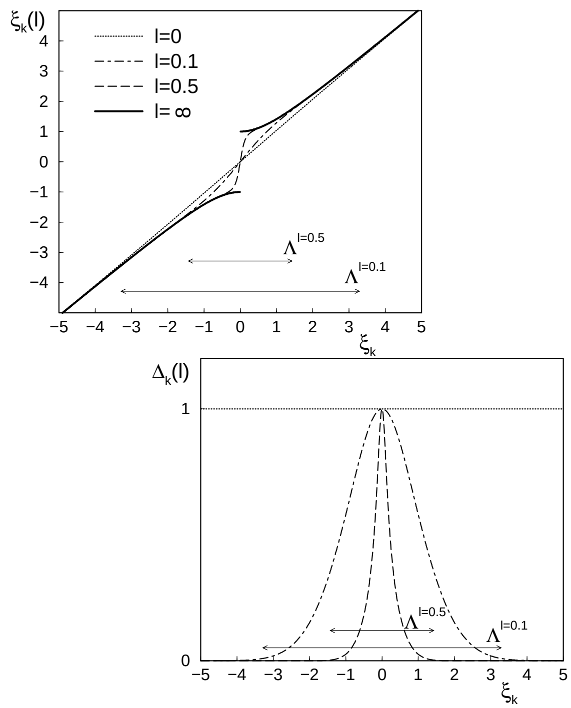

FIG. 1: Variation of $\Delta_{\mathbf{k}}(l)$ (panel on the right) and $\xi_{\mathbf{k}}(l)$ (panel on the left) versus a distance from the Fermi surface $\xi_{\mathbf{k}_F} = 0$. We assumed a constant parameter $\Delta_{\mathbf{k}}(0) = \Delta$ in the model Hamiltonian (15). Energies presented in this figure are expressed in units of Δ and the flow parameter in units of Δ^{-2}. Note that for a given l the renormalizations are practically completed outside the energy window of the width $\Lambda^l = \frac{1}{\sqrt{l}}$ as marked by the arrows.

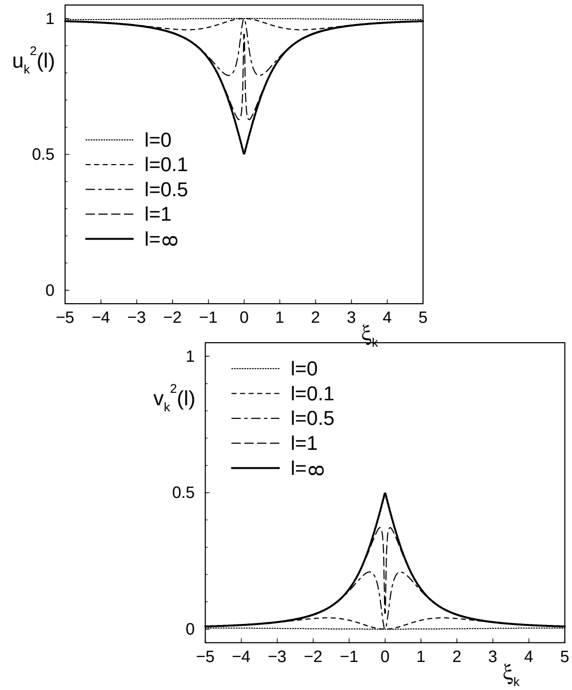

FIG. 2: The BCS coherence factors $u^2_{\mathbf{k}}(l)$ and $u^2_{\mathbf{k}}(l)$ versus the energy $\xi_{\mathbf{k}}$ for some representative values of l as indicated in the legend.

with the initial values $u_{\mathbf{k}}(0) = 1$, $v_{\mathbf{k}}(0) = 0$. From (14) we derive the following differential equations for the coefficients $u_{\mathbf{k}}(l)$, $v_{\mathbf{k}}(l)$

$$\partial_l \, u_{\mathbf{k}}(l) = 2\xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l) \, v_{\mathbf{k}}(l), \tag{23}$$

$$\partial_l \, v_{\mathbf{k}}(l) = -2\xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l) \, u_{\mathbf{k}}(l). \tag{24}$$

It can be easily checked that equations (23, 24) lead to the following invariance of l-dependent coefficients $|u_{\mathbf{k}}(l)|^2 + |v_{\mathbf{k}}(l)|^2 = 1$. This invariance assures that the fermion anticommutation relations are obeyed for arbitrary level of the transformation $\left\{\hat{c}_{\mathbf{k}\sigma}(l), \hat{c}^\dagger_{\mathbf{k}'\sigma'}(l)\right\} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'}$.

We have previously shown [12] that the asymptotic $l = \infty$ values of the coefficients $u_{\mathbf{k}}(l)$ and $v_{\mathbf{k}}(l)$ coincide with the usual BCS factors

$$u^2_{\mathbf{k}}(\infty) = \frac{1}{2}\left[1 + \frac{\xi_{\mathbf{k}}}{\sqrt{(\xi_{\mathbf{k}})^2 + |\Delta_{\mathbf{k}}|^2}}\right] \tag{25}$$

$$u_{\mathbf{k}}(\infty)v_{\mathbf{k}}(\infty) = \frac{\Delta_{\mathbf{k}}}{2\sqrt{(\xi_{\mathbf{k}})^2 + |\Delta_{\mathbf{k}}|^2}} \tag{26}$$

and $v^2_{\mathbf{k}}(\infty) = 1 - u^2_{\mathbf{k}}(\infty)$. In figure 2 we show the l-dependent factors $u^2_{\mathbf{k}}(l)$, $v^2_{\mathbf{k}}(l)$ versus the energy measured from the Fermi surface. We can notice that for arbitrary l the asymptotic values are reached by all fermion states located outside the energy cut-off $\Lambda^l = \frac{1}{\sqrt{l}}$.

The complete single particle excitation spectrum is made of the particle and hole contributions. The corresponding spectral function is found [12] to be

$$A(\mathbf{k},\omega) = u^2_{\mathbf{k}}(\infty)\delta(\omega - \xi_{\mathbf{k}}(\infty)) + v^2_{\mathbf{k}}(\infty)\delta(\omega + \xi_{\mathbf{k}}(\infty)) \tag{27}$$

In a straightforward manner we can arrive at the following expressions for the average occupancy of $\mathbf{k}$ momentum and for the order parameter

$$\langle \hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}_{\mathbf{k}\uparrow}\rangle = \frac{1}{2}\left[1 - \frac{\xi_{\mathbf{k}}}{\xi_{\mathbf{k}}(\infty)}\tanh\frac{\xi_{\mathbf{k}}(\infty)}{2k_BT}\right], \tag{28}$$

$$\langle \hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\uparrow}\rangle = \frac{\Delta_{\mathbf{k}}}{2\xi_{\mathbf{k}}(\infty)}\tanh\frac{\xi_{\mathbf{k}}(\infty)}{2k_BT}. \tag{29}$$

The Bogoliubov spectrum (20) together with the expectation values expressed in equations (28,29) reproduce the rigorous solution for the bilinear Hamiltonian (15).

V. PAIRING IN THE STRONGLY CORRELATED FERMION SYSTEM

The two particle interactions in (1) or in the reduced Hamiltonian (2) can be canceled out exactly by introducing the Hubbard-Stratonovich fields Φ, Φ^*. Skipping a detailed derivation we assign here the following effective

Hamiltonian to the resulting fermion and boson degrees of freedom [14]

$$\hat{H} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} \left(E_{\mathbf{q}} - 2\mu\right) \hat{b}^{\dagger}_{\mathbf{q}} b_{\mathbf{q}} + \frac{1}{\sqrt{N}} \sum_{\mathbf{k},\mathbf{q}} v_{\mathbf{k},\mathbf{q}} \left[\hat{b}^{\dagger}_{\mathbf{q}} \hat{c}_{-\mathbf{k},\downarrow} \hat{c}_{\mathbf{k}+\mathbf{q},\uparrow} + \text{h.c.}\right] \quad (30)$$

In the simplest step one can study (30) on the level of saddle point approximation when boson fields are replaced by constant number. This is a mean field approximation and formally it is equivalent to linearization of the boson-fermion term via

$$\hat{b}^{\dagger}_{\mathbf{q}} \hat{c}_{-\mathbf{k},\downarrow} \hat{c}_{\mathbf{k}+\mathbf{q},\uparrow} \simeq \langle \hat{b}^{\dagger}_{\mathbf{q}} \rangle \hat{c}_{-\mathbf{k},\downarrow} \hat{c}_{\mathbf{k}+\mathbf{q},\uparrow} + \hat{b}^{\dagger}_{\mathbf{q}} \langle \hat{c}_{-\mathbf{k},\downarrow} \hat{c}_{\mathbf{k}+\mathbf{q},\uparrow} \rangle - \langle \hat{b}^{\dagger}_{\mathbf{q}} \rangle \, \langle \hat{c}_{-\mathbf{k},\downarrow} \hat{c}_{\mathbf{k}+\mathbf{q},\uparrow} \rangle. \quad (31)$$

When substituting (31) to the Hamiltonian (30) the fermion and boson subsystems become decoupled from one another. The physical fermion part acquires then the bilinear structure (15) with the parameter $\Delta_{\mathbf{k}} = v_{\mathbf{k},\mathbf{0}} \langle \hat{b}_{\mathbf{q}=\mathbf{0}} \rangle$. The mean field properties of the model (30) have been summarized in the review paper [15].

In this section we show how to go beyond the mean field scenario using the flow equation method. Our main strategy is to design a continuous canonical transformation $\hat{U}(l)$ which, step by step, dismantles the fermion from boson degrees of freedom. Technical remarks concerning such transformation has been given in our previous work [16] where we constrained the Hamiltonian into $\hat{H} = \hat{H}_0 + \hat{H}_{B-F}$ with $\hat{H}_0 = \hat{H}_F + \hat{H}_B$ denoting the independent fermion and boson contributions and $\hat{H}_{B-F}$ corresponding to their interaction. We have followed the idea proposed by Wegner [7] by choosing $\hat{\eta}(l) = [\hat{H}_0(l), \hat{H}_{B-F}(l)]$ which yields

$$\hat{\eta}(l) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k},\mathbf{q}} \left[E_{\mathbf{q}}(l) - \varepsilon_{-\mathbf{k}}(l) - \varepsilon_{\mathbf{k}+\mathbf{q}}(l)\right] \times \left(v_{\mathbf{k},\mathbf{q}}(l) \hat{b}^{\dagger}_{\mathbf{q}} \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}+\mathbf{k}\uparrow} - \text{h.c.}\right). \quad (32)$$

All the l-dependent quantities have been determined by us selfconsistently using the iterative numerical Runge Kutta algorithm to solve the set of differential flow equations (16-21) presented in the Ref. [16]. We studied a situation with the fixed number of particles in the system $n_{tot} = \sum_{\mathbf{k},\sigma} \langle \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} \rangle + 2 \sum_{\mathbf{q}} \langle \hat{b}^{\dagger}_{\mathbf{q}} \hat{b}_{\mathbf{q}} \rangle$. Among the important physical effects obtained for the fixed point $l \to \infty$ we could point out that:

(c) boson particles (which can be thought as the fermion pairs) acquire a finite mobility owing to the interaction with the itinerant fermions,

(b) fermions in turn are affected by the boson particles and this effect shows up by a loss of the single particle states near the Fermi surface (such depletion of the density of states is often called in the literature as pseudogap),

(c) in addition to the pseudogap there appears a resonant scattering between fermions [17].

Let us remark that the resonant-type scattering processes have been known for a long time in the nuclear physics. Such Feshbach mechanism is nowadays widely explored experimentally and theoretically for the atomic gasses cooled to ultralow temperatures enabling observation of such quantum phenomena like the Bose Einstein condensation (BEC) and ultimately also the atomic superfuidity.

Since fermion states located near the Fermi energy get combined with the boson species it is natural to expect that the single and two-particle properties are going to affect each other. We studied systematically their interplay within the flow equation method. To derive the single particle spectrum we had to transform the annihilation $\hat{c}_{\mathbf{k}\sigma}$ and creation $\hat{c}^{\dagger}_{\mathbf{k}\sigma}$ operators using (32). From the flow equation (14) we derived the following generalized Bogoliubov transformation [18]

$$\begin{pmatrix} \hat{c}_{\mathbf{k}\uparrow}(l) \\ \hat{c}^{\dagger}_{-\mathbf{k}\downarrow}(l) \end{pmatrix} = \begin{pmatrix} u_{\mathbf{k}}(l) \\ -v^{*}_{\mathbf{k}}(l) \end{pmatrix} \hat{c}_{\mathbf{k}\uparrow} + \begin{pmatrix} v_{\mathbf{k}}(l) \\ u^{*}_{\mathbf{k}}(l) \end{pmatrix} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} + \frac{1}{\sqrt{N}} \sum_{\mathbf{q}\neq\mathbf{0}} \left[\begin{pmatrix} p_{\mathbf{k},\mathbf{q}}(l) \\ r^{*}_{\mathbf{k},\mathbf{q}}(l) \end{pmatrix} \hat{b}^{\dagger}_{\mathbf{q}} \hat{c}_{\mathbf{q}+\mathbf{k}\uparrow} + \begin{pmatrix} r_{\mathbf{k},\mathbf{q}}(l) \\ -p^{*}_{\mathbf{k},\mathbf{q}}(l) \end{pmatrix} \hat{b}_{\mathbf{q}} \hat{c}^{\dagger}_{\mathbf{q}-\mathbf{k}\downarrow} \right]. \quad (33)$$

In consequence the single particle spectral function was found to have a different structure than the usual BCS result (27). For temperatures $T < T_c$ it consists of two narrow quasiparticle peaks at energies $\omega = \pm\sqrt{(\varepsilon_{\mathbf{k}} - \mu)^2 + \Delta_{sc}^2}$ and a certain amount of the damped states forming an incoherent background outside the superconducting gap (see the left panel in figure 3). When traversing the critical temperature T_c to a normal state the gaped Bogoliubov-type spectrum seems to be preserved, however for increasing temperature the *shadow branch* becomes more and more damped (see the right h.s. panel in figure 3). Physically it means that fermion pairs have no longer an infinite life-time. Finally, for temperatures exceeding a certain characteristic value T^* the Bogoliubov modes are completely gone and there remains only a single peak at the renormalized energy $\xi_{\mathbf{k}}(\infty)$ [18].

To check a direct impact of the above mentioned behavior on the pair correlations we investigated the following correlation function

$$\sum_{\mathbf{k},\mathbf{k}'} \left\langle \hat{c}_{-\mathbf{k}\downarrow}(t) \hat{c}_{\mathbf{q}+\mathbf{k}\uparrow}(t) \;\; \hat{c}^{\dagger}_{\mathbf{q}+\mathbf{k}'\uparrow}(t') \hat{c}^{\dagger}_{-\mathbf{k}'\downarrow}(t') \right\rangle. \quad (34)$$

From the flow equation (14) for the pair operators $\sum_{\mathbf{k}} \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{q}+\mathbf{k}\uparrow}$ we obtained the corresponding spectral function

$$A_{pair}(\mathbf{q}, \omega) = \mathcal{N}_{\mathbf{q}} \, \delta\left(\omega - \tilde{E}_{\mathbf{q}}\right) + \mathcal{A}^{inc}_{\mathbf{k}}(\omega). \quad (35)$$

It contains the quasiparticle peak at $\omega = \tilde{E}_{\mathbf{q}}$ and the incoherent background $\mathcal{A}^{inc}_{\mathbf{k}}(\omega)$. For $T < T_c$ the quasi-

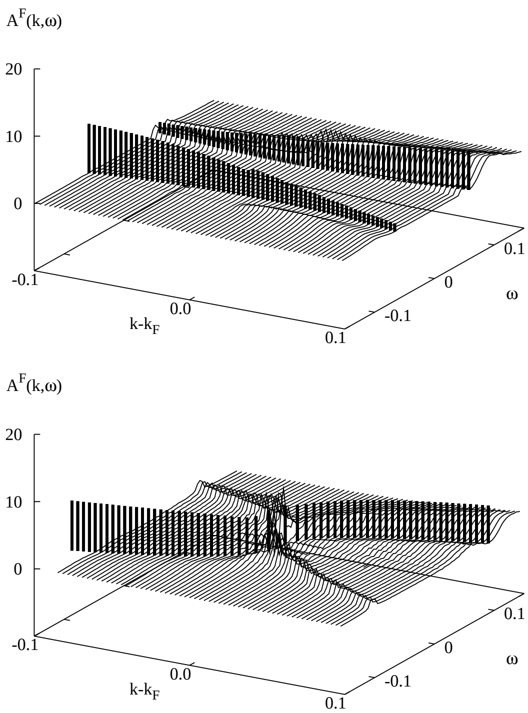

FIG. 3: The effective single particle spectrum of fermions $A^F(\mathbf{k},\omega)$ for the superconducting state (panel on the left h.s.) and in the normal state near the critical temperature $T > T_c$ (panel on the right h.s.).

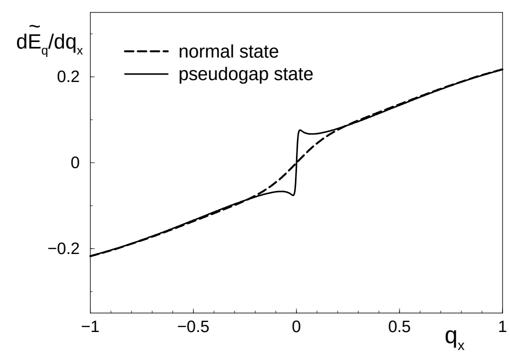

FIG. 4: Derivative of the fermion pair dispersion $d\tilde{E}_{\mathbf{q}}/dq_x$ (where $\tilde{E}_{\mathbf{q}} \equiv E_{\mathbf{q}}(\infty)$) for the normal (the dashed line) and for the pseudogap state close to T_c (the solid curve). Note a tendency for a qualitative changeover from the parabolic to linear relation of $E_{\mathbf{q}}(\infty)$.

particle peak is well separated from the incoherent background and, in the limit $\mathbf{q} \rightarrow \mathbf{0}$, has the important sound-wave dispersion $\tilde{E}_{\mathbf{q}} = c\,|\mathbf{q}|$ (like the collective sound-wave branch in the superfluid state of ^{4}He). Unfortunately in the case of charged fermions such as the conduction band electrons this mode is usually pushed to the huge plasma frequency because of the strong Coulomb repulsion. For the electrically neutral objects this *Goldstone mode* is a hallmark of the symmetry broken phase.

Above the transition temperature and close to T_c there are still some residual collective features possible to observe. At small momenta the qusiparticle peak $\omega = \tilde{E}_{\mathbf{q}}$ overlaps with the incoherent background therefore the long wavelength limit is not suitable for appearance of the collective effects (strictly speaking for $\mathbf{q} \rightarrow \mathbf{0}$ the Goldstone mode is replaced by a parabolic dispersion). A remnant of the Goldstone mode splits off from the incoherent background for finite momenta above a certain critical value $\mathbf{q}_{crit}$. This is illustrated in figure 4.

Collective features seen in the pair-pair correlations (or in the density-density correlations) tell us directly about a presence or absence of the long range order which is necessary for the onset superconductivity/superfluidity. As a matter of fact the single particle properties are then ill defined (at least in a vicinity of the Fermi energy). Instead of single fermions one should rather think about fermion pairs as good quasiparticles. Although the fermion pairs are necessary for superconudctivity/superfluidity the other way around this is not valid. It has been shown by Eagles [19], Nozieres and Schmitt-Rink [20] and by a number of other authors that value of the binding energy (magnitude of the gap in the single particle spectrum) does not scale linearly with the superfluid phase stiffness n_s which determines the transition temperature T_c. Our work indeed confirms that existence of the preformed (i.e. incoherent) fermion pairs is a natural expectation while approaching T_c from above. Such type of situation can be encountered in the high T_c cuprate superconductors where the strong quantum fluctuations are driven by the unscreened Coulomb repulsion between electrons and by the reduced dimensionality of CuO_2 planes [21]. Pseudogap have been also unambiguously observed in the ultracold fermion atoms close to the unitarity limit (i.e. on the Feshbach resonance) [22].

VI. SUMMARY

We have presented the method of continuous unitary transformation originating from a general scheme of the RG scaling procedure. This non-perturbative technique overcomes usual problems of the standard RG methods in application to the symmetry broken states. We have illustrated it on the exactly solvable case of the bilinear Hamiltonian. This new method is moreover capable to study the possible feedback effects between the *fast* and *slow* modes treating both sectors simultaneously throughout the whole continuous transformation.

Applying this method to the strongly interacting fermion system we have shown that formation of the fermion pairs need not be accompanied by the transition to superfluid/superconducting state. Strong quantum fluctuations can suppress the long-range coherence

(ordering) so that effectively the fermion pairs exist even in the normal state above T_c. Such preformed fermion pairs could be observed experimentally by e.g. probing the single particle spectrum (using the STM or ARPES spectroscopies) or in measurements of the correlations between pairs (via any experimental technique sensitive to the pair susceptibility).

Acknowledgements

This work is partly supported by the KBN grant 2P03B06225. We acknowledge fruitful discussions with J. Ranninger and K.I. Wysokiński.

[1] K.G. Wilson and J. Kogut, Rev. Rep. C **12**, 75 (1974); S. Ma, Rev. Mod. Phys. **45**, 589 (1973); M.E. Fisher, Rev. Mod. Phys. **46**, 597 (1974); J.A. Hertz, Phys. Rev. B **14**, 1165 (1976).

[2] M. Gell-Mann, F.E. Low, Phys. Rev. **95**, 1300 (1954); L.P. Kadanoff, Physics **2**, 263 (1967).

[3] M.E. Fisher, Rev. Mod. Phys. **70**, 653 (1998); W. Metzner, C. Castellani, C. di Castro, Adv. Phys. **47**, 317 (1998); R. Shankar, Rev. Mod. Phys. **66**, 129 (1994).

[4] R. Gersch, J. Reiss and C. Honerkamp, cond-mat/0609520.

[5] M. Salmhofer, C. Honercamp, W. Metzner, O. Lauscher, Prog. Theor. Phys. **112**, 943 (2004); A. Katanin, Phys. Rev. B **70**, 115109 (2004); C. Honercamp and M. Salmhofer, Prog. Theor. Phys. **113**, 1145 (2005).

[6] F. Schütz, L. Bartosch and P. Kopietz, Phys. Rev. B **72**, 035107 (2005); F. Schütz and P. Kopietz, cond-mat/0512027.

[7] F. Wegner, Annalen der Physik (Leipzig) **3**, 77 (1994).

[8] S.D. Głazek and K.G. Wilson, Phys. Rev. D **49**, 4214 (1994).

[9] F. Wegner, J. Phys. A: Math. Gen. **39**, 8221 (2006).

[10] S. Kehrein, *The flow equation approach to many-particle systems*, Springer Tracts in Modern Physics **217**, (2006).

[11] N.N. Bogoliubov, Sov. Phys. JETP **7**, 41 (1948).

[12] T. Domański, cond-mat/0602236.

[13] G. Stelter, *Zur Lösung von Flussgleichungen für einfache Systeme*, diploma thesis no. 3291 (Heidelberg, 1996) unpublished.

[14] J. Ranninger, S. Robaszkiewicz, Physica B **135**, 468 (1985); S. Robaszkiewicz, R. Micnas, J. Ranninger, Phys. Rev. **B 36**, 180 (1987); J. Ranninger, R. Micnas, S. Robaszkiewicz, Ann. Phys. Fr. **13**, 455 (1988); R. Friedberg, T.D. Lee, Phys. Rev. B **40**, 423 (1989); R. Friedberg, T.D. Lee, H.C. Ren, Phys. Lett. A **152**, 417 (1991).

[15] R. Micnas, J. Ranninger, S. Robaszkiewicz, Rev. Mod. Phys. **62**, 113 (1990).

[16] T. Domański and J. Ranninger, Phys. Rev. B **63**, 134505 (2001).

[17] T. Domański, Phys. Rev. A **68**, 013603 (2003).

[18] T. Domanski, J. Ranninger, Phys. Rev. Lett. **91**, 255301 (2003); Phys. Rev. B **70**, 184503 (2004); Phys. Rev. B **70**, 184513 (2004).

[19] D.M. Eagles, Phys. Rev. **186**, 456 (1969).

[20] P. Nozières, S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985).

[21] P.A. Lee, N. Nagaosa, X.G. Wen, Rev. Mod. Phys. **78**, 17 (2006).

[22] Q. Chin, J. Stajic, S. Tan, K. Levin, Phys. Rep. **412**, 1 (2005).